Do Deep Learning Models Have Too Many Parameters? An Information Theory Viewpoint

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Abstract

Deep learning models often have more parameters than observations, and still perform well. This is sometimes described as a paradox. In this work, we show experimentally that despite their huge number of parameters, deep neural networks can compress the data losslessly *even when taking the cost of encoding the parameters into account*. Such a compression viewpoint originally motivated the use of *variational methods* in neural networks [HV93, Sch97]. However, we show that these variational methods provide surprisingly poor compression bounds, despite being explicitly built to minimize such bounds. This might explain the relatively poor practical performance of variational methods in deep learning. Better encoding methods, imported from the Minimum Description Length (MDL) toolbox, yield much better compression values on deep networks, corroborating the hypothesis that good compression on the training set correlates with good test performance.

1 Introduction

Deep Learning has recently achieved outstanding results in many different areas [LBH15]. One of the poorly understood properties of deep models is their ability to not overfit despite their large number of parameters.

The *Cambridge Dictionary of Statistics* [ES10] defines overfitted models as *"models that contain more unknown parameters than can be justified by the data"*. This does not seem to align well with overfit as understood in machine learning, based on practical performance on a test set. Deep learning models would be rejected out of hand by traditional criteria such as AIC [Aka74] and BIC [Sch78], which heavily penalize the number of parameters.

Information theory and Minimum Description Length (MDL) provide a measure of the complexity of a model that is not directly based on its number of parameters, but on its ability to losslessly *compress* the training data (including the cost of describing the model itself). One may wonder if this criterion is better aligned with practical performance for deep learning models. This would vindicate Chaitin's hypothesis, based on Occam's razor, that "*comprehension is compression*" [Cha07]: any regularity in the data can be used identically to compress it or make predictions. This is ultimately rooted in Solomonoff's general theory of inference [Sol64] (see also, e.g., [Hut07, Sch97]): the main principle is to prefer models that correspond to the "shortest program" to produce the training data, based on its Kolmogorov complexity [LV08].

To measure compression, the problem is recast as a (lossless) data transmission problem on the training set. In particular, this protects against one of the aspects of overfitting in machine learning, namely, data memorization (which does not compress). This also amounts to estimating the *mutual information* between the inputs and labels, as studied e.g. [SZT17] in relationtionship with generalization ability.

Any parametric probabilistic model can be turned into explicit lossless compression bounds on the data *including the cost of transmitting a description of the model*, thanks to a number of techniques from the Minimum Description Length (MDL) theory [Grü07]. This will naturally penalize more complex models. In the neural network community, MDL was introduced as far back as [HV93], via variational inference, which explicitly optimizes a compression bound. However, this has not been widely adopted in deep learning, and has been found to decrease practical performance. This seems to go against Chaitin's hypothesis.

In this work, we examine Chaitin's hypothesis in the context of Deep Learning. We do not introduce new deep learning methods or architecturesr; rather, we take some already working methods and try to understand why they generalize by investigating if they can compress the training data. Our contributions are as follow:

- We confirm the fact, already observed in [Gra11] using variational codes, that standard deep learning models can represent the data *together with the model* in fewer bits than a naive encoding of the data. This holds even for datasets as small as MNIST.
- However, we find that variational inference yields surprisingly inefficient codelengths, despite explicitly being a codelength minimization method. Other methods from the MDL toolbox, such as *prequential coding*, yield substantially better codelengths that correlate better with test set performance.
- We introduce practical ways to compute efficient compression bounds with deep learning models. In particular, this provides ways to quantify the mutual information between inputs and outputs. We also introduce *self-switch*, an efficient way to improve on prequential compression bounds for models learned by an iterative algorithm like gradient descent.
- As an aside, we show that while deep learning models can fit fake labels [ZBH+17], they are not able to *compress* them: models trained using MDL-based regularization such as variational methods behave very differently on actual or fake labels and do not overfit the fake labels, successfully detecting the absence of mutual information.

2 Probabilistic Models, Compression, and Information Theory

Imagine that Alice wants to efficiently transmit some information to Bob. Alice has a dataset $\mathcal{D} = \{(x_1, y_1), ..., (x_n, y_n)\}$ where $x_1, ..., x_n$ are some inputs and $y_1, ..., y_n$ some labels. We do not assume that these data come from a "true" probability distribution. Bob also has the data $x_1, ..., x_n$, but he does not have the labels. This describes a *supervised learning* situation in which the inputs x may be publicly available, and a prediction of the labels y is needed. How can deep learning models help with data encoding? One key problem is that Bob does not necessarily know the precise, trained model that Alice is using. So some explicit or implicit transmission of the model itself is required.

We study, in turn, various methods to encode the labels y, with or without a deep learning model. Encoding the labels knowing the inputs is equivalent to estimating their mutual information (Section 2.4); this is distinct from the problem of practical network compression (Section 3.2) or from using neural networks for lossy data compression.

Our running example will be image classification on the MNIST [LBBH98] and CIFAR10 [Kri09] datasets.

2.1 Definitions and notation

Let $\mathcal{X}$ be the input space and $\mathcal{Y}$ the output (label) space. In this work, we only consider classification tasks, so $\mathcal{Y} = \{1, ..., K\}$. The dataset is $\mathcal{D} := \{(x_1, y_1), ..., (y_n, x_n)\}$. Denote $x_{k:l} := (x_k, x_{k+1}, ..., x_{l-1}, x_l)$.

We define a *model* for the supervised learning problem as a conditional probability distribution $p(y|x)$, namely, a function such that for each $x \in \mathcal{X}$, $\sum_{y \in \mathcal{Y}} p(y|x) = 1$. A *model class*, or *architecture*, is a set of models depending on some parameter θ: $\mathcal{M} = \{p_\theta, \theta \in \Theta\}$.

The *Kullback–Leibler divergence* between two distributions is

$$\mathrm{KL}(\mu \| \nu) = \mathbb{E}_{X \sim \mu}[\log_2 \frac{\mu(x)}{\nu(x)}] \tag{1}$$

.

2.2 Models and codelengths

We recall a basic result of compression theory [Sha48].

Proposition 1 (Shannon–Huffman code)**.** *Suppose that Alice and Bob have agreed in advance on a model p, and both know the inputs $x_{1:n}$. Then there exists a code to transmit the labels $y_{1:n}$ losslessly with codelength*

$$L_p(y_{1:n}|x_{1:n}) = -\sum_{i=1}^{n} \log_2 p(y_i|x_i) \tag{2}$$

up to at most one bit.

This bound is known to be optimal if the data are independent and coming from the model p [Mac03].

The one additional bit in the Shannon–Huffman code is incurred only once for the whole dataset [Mac03]. With large datasets this is negligible. Thus, from now on we will systematically omit the $+1$ as well as admit non-integer codelengths [Grü07]. We will use the terms *codelength* or *compression bound* interchangeably.

This bound is exactly the categorical *cross-entropy loss* evaluated on the model p. Hence, trying to minimize the description length of the outputs over the parameters of a model class is equivalent to minimizing the usual classification loss.

Here we do not deal with the practical implementation of compression algorithms: we only care about the theoretical *bit length* of their associated encodings. We are interested in measuring the amount of information contained in the data, the mutual information between input and output, and how it is captured by the model. Formally, a codelength can be defined as any function L that satisfies the Kraft inequality $\sum_y 2^{-L(y)} \leq 1$ [Mac03]. This implies the existence of an actual encoding with codelength L up to ± 1. Thus, we will directly work with codelength functions.

An obvious limitation of the bound (2) is that Alice and Bob both have to know the model p in advance. This is problematic if the model must be learned from the data.

2.3 Uniform encoding

The uniform distribution $p_{\text{unif}}(y|x) = \frac{1}{K}$ over the K classes does not require any learning from the data, thus no additional information has to be transmitted. Using this *uniform encoding* in (2) results in the codelength

$$L^{\text{unif}}(y_{1:n}|x_{1:n}) = n \log_2 K \tag{3}$$

This uniform encoding will be a sanity check against which to compare the other encodings in this text.

For MNIST, the uniform encoding cost is $60000 \times \log_2 10 = 199\,\text{kbits}$. For CIFAR, the uniform encoding cost is $50000 \times \log_2 10 = 166\,\text{kbits}$.

2.4 Mutual information between inputs and outputs

Intuitively, the only way to beat a trivial encoding of the outputs is to use the mutual information (in a loose sense) between the inputs and outputs.

This can be formalized as follows. Assume that the inputs and outputs follow a "true" joint distribution $q(x, y)$. Then any transmission method with codelength L satisfies [Mac03]

$$\mathbb{E}_q[L(y|x)] \geq H(y|x) \tag{4}$$

Therefore, the gain (per data point) between the codelength L and the trivial codelength $H(y)$ is

$$H(y) - \mathbb{E}_q[L(y|x)] \leq H(y) - H(y|x) = I(y; x) \tag{5}$$

the mutual information between inputs and outputs [Mac03].

Thus, the gain of *any* codelength compared to the uniform code is limited by the amount of mutual information between input and output. (This bound is reached with the true model $q(y|x)$.) Any successful compression of the labels is, at the same time, a direct estimation of the mutual information between input and output. The latter is the central quantity in the Information Bottleneck approach to deep learning models [SZT17].

Note that this still makes sense without assuming a true underlying probabilistic model, by replacing the mutual information with the "absolute" mutual information $K(y) - K(y|x)$ based on Kolmogorov complexity K [LV08].

3 Compression Bounds via Deep Learning

Let us introduce different compression methods based on deep learning models, taken from the MDL toolbox.

3.1 Two-part encodings

In a typical situation, Alice and Bob might first agree on a model class (such as "neural networks with two layers and 1,000 neurons per layer"). However, Bob does not have access to the labels, so Bob cannot train the parameters of the model. Therefore, if Alice wants to use such a parametric model, the parameters themselves would have to be transmitted.

Such codings in which Alice first transmits the parameters of a model, then encodes the data using this parameter, have been called *two-part codes* [Grü07].

Table 1: **Compression bounds via Deep Learning.** Compression bounds given by different codes on two datasets, MNIST and CIFAR10. The *Codelength* is the number of bits necessary to send the labels to someone who already has the inputs. This codelength *includes* the description length of the model. The *compression ratio* for a given code is the ratio between its codelength and the codelength of the uniform code. The *test accuracy* of a model is the accuracy of its predictions on the test set. For 2-part and network compression codes, we report results from [HMD15] and [XYZL17] . The values in the table for these codelengths and compression ratio are upper bounds, since it only takes into account the codelength of the weights, and not the codelength of the data encoded with the model. For variational, prequential, switch and self-switch codes, we selected the model and hyperparameters providing the best compression bound.

	Codelength (kbits)	Compression Ratio	Test Accuracy
Uniform	199	1.	10%
float32 2-part	$>$ 8.6Mb	$>$ 45.	98.4%
Network compr.	$>$ 400	$>$ 1.	98.4%
Variational	23.9	0.12	95.5%
Prequential	**4.10**	**0.02**	**99.5%**
Switch	**4.05**	**0.02**	**99.5%**
Self-Switch	**4.05**	**0.02**	**99.5%**

(a) Compression bounds for MNIST Dataset

	Codelength (kbits)	Compression Ratio	Test Accuracy
Uniform	166	1.	10%
float32 2-part	$>$ 428Mb	$>$ 2500.	**92.9%**
Network compr.	$>$ 14Mb	$>$ 83.	**93.3%**
Variational	89.0	0.54	61,6%
Prequential	45.3	0.27	**93.3%**
Switch	**34.6**	**0.21**	**93.3%**
Self-Switch	**34.9**	**0.21**	**93.3%**

(b) Compression bounds for CIFAR Dataset

Definition 1 (Two-part codes)**.** Assume that Alice and Bob have first agreed on $(p_\theta)_{\theta\in\Theta}$ a model class. Let $L_{\text{param}}(\theta)$ be any encoding scheme for parameters $\theta \in \Theta$. Let θ^* be any parameter. The corresponding *two-part codelength* is

$$L^{\text{2-part}}_{\theta^*}(y_{1:n}|x_{1:n}) := L_{\text{param}}(\theta^*) + L_{p_{\theta^*}}(y_{1:n}|x_{1:n}) \tag{6}$$

$$= L_{\text{param}}(\theta^*) - \sum_{i=1}^{n} \log_2 p_{\theta^*}(y_i|x_i) \tag{7}$$

There exist different codes L_{param} for θ. One of them is the standard float32 binary encoding for θ, for which

$$L_{\text{param}}(\theta) = 32 \dim(\theta). \tag{8}$$

In deep learning, two-part codes are widely inefficient and much worse than the uniform encoding [Gra11]. For a model with 1 million parameters, the two-part code with float32 binary encoding will amount to 32 Mbits, or 200 times the uniform encoding on CIFAR10.

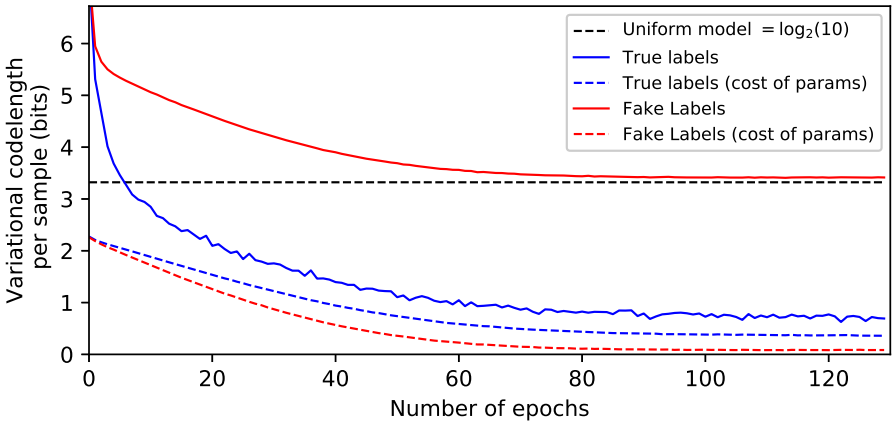

Figure 1: **Detecting fake labels with variational learning.** We train a Deep Neural Network model on MNIST with true and fake labels, both with and without variational learning. The model is an MLP with 3 hidden layers of size 200, with RELU units. With ordinary training, the model is able to overfit random labels, though training is slower than with the true labels. The plot shows the effect of using variational learning instead, and reports the variational objective (encoding cost of the training data), on true and fake labels. We isolated the contribution from parameter encoding in the total loss (KL term in (9)). With true labels, the encoding cost is below the uniform encoding, and half of the description length is information contained in the weights. With fake labels, on the contrary, the encoding cost converges to a uniform random model, with no information contained in the weights: there is no mutual information between inputs and outputs.

3.2 Practical network compression and two-part codes

The practical encoding of trained models is a well-developed research topic, e.g., for use on small devices such as cell phones. Such encodings can be considered as a two-part code using a clever code for θ instead of encoding every parameter on 32 bits.

Different strategies exist, such as training a *student layer* to approximate a well-trained network [BC14, RBK+15], or using a pipeline involving retraining, pruning, and quantization of the model weights [HMD15, HPTD15, SZ14, LUW17, SLM16, UMW17].

Still, the resulting codelengths (for compressing the labels given the data) are way above the uniform compression bound for image classification. With the different methods cited above, the network weights are compressed by a factor less than 100. Even for a relatively small network with 1 million parameters, encoding the compressed network will cost 320 kbits, already twice above the uniform codelength for MNIST or CIFAR.

3.3 Variational Code: Networks with Random Weights

Another strategy for encoding weights with a limited precision is to represent these weights by random variables: the uncertainty on θ represents the precision with which θ is transmitted. The *variational code* turns this into an explicit encoding scheme, thanks to the *bits-back* argument [HV04]. This idea was introduced as a way to compute codelength bounds with neural networks [HV93], but is now more used as a regularization technique [BCKW15]. This method yields the following codelength.

Definition 2 (Variational code)**.** Assume that Alice and Bob have agreed on a model class $(p_\theta)_{\theta\in\Theta}$ and a prior α over Θ. Then for any distribution β over Θ, there exists an encoding with the following codelength:

$$L_\beta^{\mathrm{var}}(y_{1:n}|x_{1:n}) = \mathrm{KL}\,(\beta\|\alpha) + \mathbb{E}_{\theta\sim\beta}[L_{p_\theta}(y_{1:n}|x_{1:n})] \tag{9}$$

$$= \mathrm{KL}\,(\beta\|\alpha) + \mathbb{E}_{\theta\sim\beta}\left[\sum_{i=1}^{n} -\log_2 p_\theta(y_i|x_i)\right] \tag{10}$$

The compression bound given by the variational code can be minimized over β by choosing a parametric model class $(\beta_\phi)_{\phi\in\Phi}$, and minimizing (9) over ϕ. A common model class for β is the set of multivariate Gaussian distributions $\{\mathcal{N}(\mu,\Sigma), \mu\in\mathbb{R}^d, \Sigma \text{ diagonal}\}$, and μ and Σ can be optimized with a stochastic gradient descent algorithm [Gra11, KTR+17]. Σ can be interpreted as the precision with which the parameters are encoded.

The variational bound L_β^{var} is an upper bound for the Bayesian description length bound of the Bayesian model p_θ with parameter θ and prior α. Considering the Bayesian distribution of y,

$$p_{\mathrm{Bayes}}(y_{1:n}|x_{1:n}) = \int_{\theta\in\Theta} p_\theta(y_{1:n}|x_{1:n})\alpha(\theta)d\theta, \tag{11}$$

then Proposition 1 provides an associated code via (2) with model p_{Bayes}:

$$L^{\mathrm{Bayes}}(y_{1:n}|x_{1:n}) = -\log_2 p_{\mathrm{Bayes}}(y_{1:n}|x_{1:n}) \tag{12}$$

Then, for any β we have [Gra11]

$$L_\beta^{\mathrm{var}}(y_{1:n}|x_{1:n}) \geq L^{\mathrm{Bayes}}(y_{1:n}|x_{1:n}) \tag{13}$$

with equality if and only if β is equal to the Bayesian posterior $p_{\mathrm{Bayes}}(\theta|x_{1:n}, y_{1:n})$. Variational methods can be seen as a way to approximate Bayesian inference, since this Bayesian posterior is often intractable.

We computed practical compression bounds with variational methods on MNIST and CIFAR10. Neural networks that give the best variational compression bounds appear to be smaller than networks trained the usual way. We tested various fully connected networks and convolutional networks: the models that gave the best variational compression bounds were small fully connected networks with 2 hidden layers of size 256. The random weights for β are also used at test time; the posterior average parameter has also been used and improves performance a bit [BCKW15], but this does not test Chaitin's hypothesis. Moreover, we selected the best model in Table 1 based on compression, not test accuracy. This results in a drop of test accuracy with respect to other settings, directly challenging the link between compression and test accuracy.

On MNIST, this provides a codelength of the labels (knowing the inputs) of 24.1 kbits, i.e., a compression ratio of 0.12. The model which reached this score achieved 95.5% classification accuracy on the test set.

On CIFAR, we obtained a codelength of 89.0 kbits, i.e., a compression ratio of 0.53. The model which reached this score achieved 61.6% classification accuracy on the test set.

We can make two observations. First, choosing the model class which minimizes variational codelength selects smaller deep learning models than a cross-validation procedure. Second, the selected model has low classification accuracy on the test set on MNIST and CIFAR, compared to models trained without variational method. This seems to agree with a common criticism of Bayesian and MDL methods as too conservative as model selection procedures: these methods are consistent, but in many settings, their convergence rates are a factor $O(\log n)$ slower than leave-one-out cross-validation or AIC criterion [RSY92, FG94, BY99, Grü07].

Algorithm 1 Prequential encoding

Input: data $x_{1:n}, y_{1:n}$, timesteps $1 = t_0 < t_1 < ... < t_S = n$
Alice transmits the random seed ω (if using stochastic optimization).
Alice encodes $y_{1:t_1}$ with the uniform code. This costs $t_1 \log_2 K$ bits. Bob decodes $y_{1:t_1}$.
for $s = 1$ **to** $S - 1$ **do**
  Alice and Bob both compute $\hat{\theta}_s = \hat{\theta}(x_{1:t_s}, y_{1:t_s}, \omega)$.
  Alice encodes $y_{t_s+1:t_{s+1}}$ with model $p_{\hat{\theta}_s}$. This costs $-\log_2 p_{\hat{\theta}_s}(y_{t_s+1:t_{s+1}}|x_{t_s+1:t_{s+1}})$ bits
  Bob decodes $y_{t_s+1:t_{s+1}}$
end for

We will use another coding procedure to show that deep neural models which generalize well also compress well.

Still, variational methods are an efficient regularization method against overfitting. Convolutional neural networks commonly used for image classification tasks are able to reach 100% accuracy on random labels [ZBH+17], even though there is no mutual information between the inputs and the labels. However, reproducing this experiment with codelength as the loss function (instead of unregularized classification loss), we observe that deep neural models are able to learn the true labels, but that on fake labels the model just converges to the random uniform model as expected. See Figure 1 for details.

3.4 Prequential or Online code

The prequential (or online) code is a way to encode both the model and the labels without *directly* encoding the weights, founded on the *prequential approach to statistics* [Daw84], by using *prediction strategies*. We call p a *prediction strategy* for predicting the labels in $\mathcal{Y}$ knowing the inputs in $\mathcal{X}$ if for all k, $p(y_{k+1}|x_{1:k+1}, y_{1:k})$ is a conditional model; namely, any strategy for predicting the $k+1$- label after already having seen k input-output pairs. In particular, such a model may be *learned* using the k first data samples. Any prediction strategy defines a model on the whole dataset:

$$p^{\text{preq}}(y_{1:n}|x_{1:n}) = p(y_1|x_1) \cdot p(y_2|x_{1:2}, y_1) \cdot \ldots \cdot p(y_n|x_{1:n}, y_{1:n-1}) \tag{14}$$

Let $(p_\theta)_{\theta \in \Theta}$ be a deep learning model. We assume that we have a learning algorithm which outputs, from any number of data samples $(x_{1:k}, y_{1:k})$, a weight vector $\hat{\theta}(x_{1:k}, y_{1:k})$. Even though the encoding procedure is called "online", it does not mean that only the most recent sample is used to update the parameter $\hat{\theta}$: the optimization procedure $\hat{\theta}$ can be any deterministic or stochastic optimization technique using all the previous samples $(x_{1:k}, y_{1:k})$, only requiring that the algorithm has an explicit stopping criterion.

This technique encodes the data in an incremental way: at each step k, $\hat{\theta}(x_{1:k}, y_{1:k})$ is used to predict y_{k+1}. In practice, the learning procedure $\hat{\theta}$ may only reset and retrain the network at certain timesteps. We choose timesteps $1 = t_0 < t_1 < ... < t_S = n$, and we encode the data by blocks, always using the model learned from the already transmitted data (Algorithm 1).

In practice, the optimization procedure for neural networks may be stochastic (initial values, dropout, data augmentation...), and Alice and Bob need to make all the same random actions in order to get the same final model. A simple way to do this without sending more informations is to agree on a random seed ω (or pseudorandom numbers) beforehand, so that the random optimization procedure

$\hat{\theta}(x_{1:t_s}, y_{1:t_s})$ is a deterministic one defined by $\hat{\theta}(x_{1:t_s}, y_{1:t_s}, \omega)$. Then Bob is always able to decode the data. This code, using a uniform encoding for the first few points, yields the following description length:

Definition 3 (Prequential code)**.** Given a model p_θ and a learning algorithm $\hat{\theta}(x_{1:k}, y_{1:k})$, the *prequential* codelength is

$$L^{\text{preq}}(y_{1:n}|x_{1:n}) = t_1 \log_2 K + \sum_{s=0}^{S-1} -\log_2 p_{\hat{\theta}_{t_s}}(y_{t_s+1:t_{s+1}}|x_{t_s+1:t_{s+1}}) \tag{15}$$

where for each s, $\hat{\theta}_{t_s} = \hat{\theta}(x_{1:t_s}, y_{1:t_s})$ is the parameter learned on data samples 1 to t_s.

The model parameters are never encoded directly in this algorithm. This method takes full advantage of the model's generalization ability from the previous data to the next.

Prequential coding considerably improves compression bounds. On MNIST, we computed the description length of the labels with different networks (Appendix A.2). The best compression bound was given by a convolutional network of depth 8. It achieved a description length of 4.10 kbits, i.e., a compression ratio of 0.021, with 99.5% test set accuracy. This codelength is 6 times smaller than the variational codelength.

On CIFAR, we tested a simple multilayer perceptron (MLP), a small convolutional network (tinyCNN), and a VGG-like [SZ14] convolutional network with two different training procedures: without data augmentation and batch normalization (VGGa) [IS15], and with both of them (VGGb) (Appendix A.2). The results are in Figure 2. The best compression bound was obtained with the VGG-like network trained with data augmentation and batch normalization, achieving a codelength of 45.3 kbits, i.e., a compression ratio of 0.27, and 93% test set accuracy. This codelength is twice smaller than the variational codelength. The difference between VGGa and VGGb's codelengths also show the impact of the training procedure on compression bounds, even with a fixed architecture.

4 Switching between models against the *catch-up phenomenon*

A weakness of prequential codes is the *catch-up phenomenon*, identified by [VGD12] in the context of Bayesian model selection. Assume that Alice and Bob agree on a large architecture for sending the labels of the data knowing the inputs with the prequential code, with a non-regularized gradient descent algorithm. Then, there is a risk of overfitting during the first steps of the prequential encoding, when the model is trained with a few data samples. Consequently, the encoding cost of the first packs of data might be larger than with the uniform code. Even after the encoding cost for the next labels becomes lower, the cumulated codelength will need a lot of time to *catch-up*, and reach a compression ratio below 1. The *catch-up* phenomenon can be observed in practice when learning with neural networks: in Figure 2, the model (VGGb) needs 5,000 samples on CIFAR to reach a compression ratio < 1, even though the encoding cost per label become better than uniform after 1,000 samples.

4.1 Switching between model classes

The solution introduced by [VGD12] is to *switch* between models, to always encode a data block with the best model at that point. That way, the encoding adapts itself to the number of data samples seen.

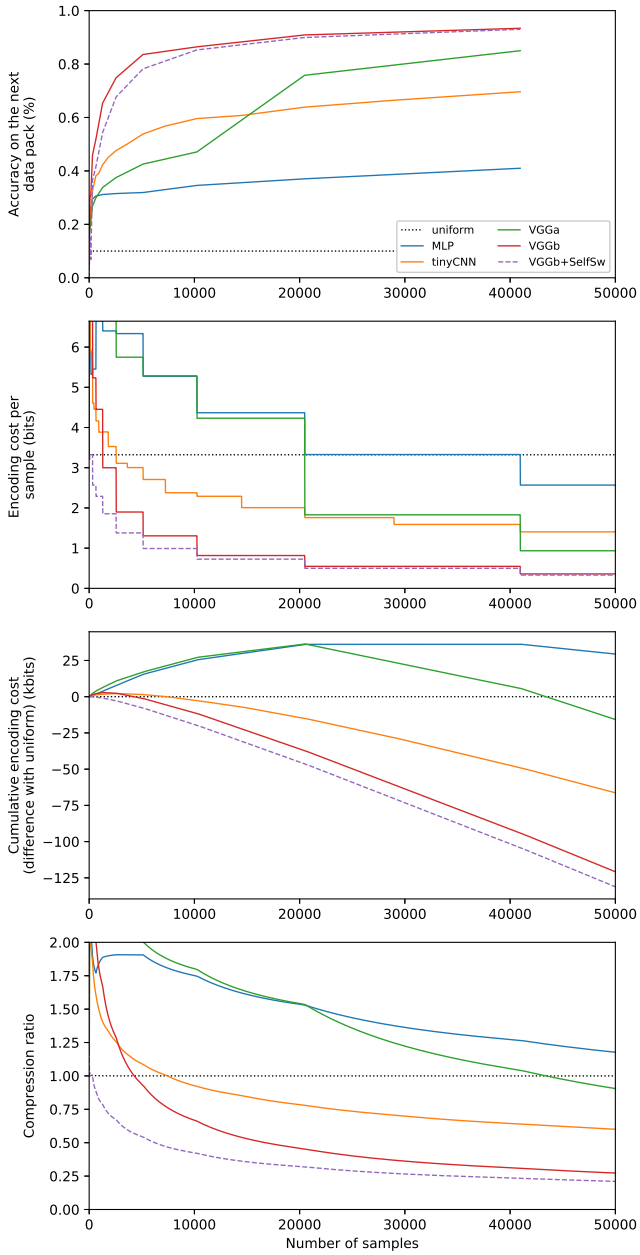

Figure 2: **Prequential and Self-Switch code results on CIFAR.** Results of prequential encoding on CIFAR with 4 different models: a small Multilayer Perceptron (MLP), a small convolutional layer (tinyCNN), a VGG-like network without data augmentation and batch normalization (VGGa) and the same VGG-like architecture but trained with data augmentation and batch normalization (VGGb) (see Appendix A.2). Performance is reported during online training, as a function of the number of samples seen so far. Top: test accuracy on a pack of data $[t_k; t_{k+1})$ given data $[1; t_k)$, as a function of t_k. Second: codelength per sample (log loss) on a pack of data $[t_k; t_{k+1})$ given data $[1; t_k)$. Third: difference between the prequential cumulated codelength on data $[1; t_k]$, and the uniform encoding. Bottom: compression ratio of the prequential code on data $[1; t_k]$.

Assume that Alice and Bob have agreed on a set of prediction strategies $\mathcal{M} = \{p^k, k \in \mathcal{I}\}$. We define the set of switch sequences, $\mathbb{S} = \{((t_1, k_1), ..., (t_L, k_L)), 1 = t_1 < t_2 < ... < t_L \ , \ k \in \mathcal{I}\}$.

Let $s = ((t_1, k_1), ..., (t_L, k_L))$ be a switch sequence. The associated prediction strategy $p_s(y_{1:n}|x_{1:n})$ uses model p^{k_i} on the time interval $[t_i; t_{i+1})$, namely

$$p_s(y_{1:i+1}|x_{1:i+1}, y_{1:i}) = p^{K_i}(y_{i+1}|x_{1:i}, y_{1:i}) \tag{16}$$

where K_i is such that $K_i = k_l$ for $t_l \leq i < t_{l+1}$. Fix a prior distribution π over switching sequences (see [VGD12] for typical examples).

Definition 4 (Switch code)**.** Assume that Alice and Bob have agreed on a set of prediction strategies $\mathcal{M}$ and a prior π over $\mathbb{S}$. The *switch code* first encodes a switch sequence s strategy, then uses the prequential code with this strategy:

$$L_s^{\mathrm{sw}}(y_{1:n}, x_{1:n}) = L_\pi(s) + L_{p_s}^{\mathrm{preq}}(y_{1:n}, x_{1:n}) \tag{17}$$

$$= -\log_2 \pi(s) - \sum_{i=1}^{n} \log_2 p^{K_i}(y_i|x_i, x_{1:i-1}, y_{1:i-1}) \tag{18}$$

where K_i is the model used by switch sequence s at time i.

We tested switching between the uniform model, a small convolutional network (tinyCNN), and a VGG-like network with two training methods (VGGa, VGGb) (Appendix A.2). On MNIST, switching between models does not make much difference. On CIFAR10, switching by taking the best model on each interval $[t_k; t_{k+1})$ saves more than 11 kbits, reaching a codelength of 34.6 kbits, and a compression ratio of 0.21. The cost $L_\pi(s)$ of encoding the switch s is negligible.

4.2 Self-Switch: Switching between versions of a model

It may be cumbersome to work with different models in parallel. Instead, for models learned by gradient descent, we may use the same architecture but with different parameter values corresponding to different gradient descent stopping times.

Let $(p_\theta)_{\theta \in \Theta}$ be a model class. Let $\hat{\theta}_t(x_{1:k}, y_{1:k})$ be the parameter obtained by some optimization procedure after t epochs of training. For instance, $t = 0$ would correspond to using an untrained model (usually close to the uniform model).

We call *self-switch code* the switch code obtained by switching among the family of models with different gradient descent stopping times t (based on the same parametric family $(p_\theta)_{\theta \in \Theta}$). In practice, this means that at each step of the prequential encoding, after having seen data $[1; t_k)$, we train the model on those data and record, at each epoch, the loss obtained on data $[t_k; t_{k+1})$. We then switch optimally between those. We incur the small additional cost of encoding the number of epochs used for training (which was limited to 10). This is a form of regularization via early stopping.

The self-switch code achieves similar compression bounds to the switch code, while storing only one network. On MNIST, there is no difference. On CIFAR, self-switch loses only 300 bits (0.006 bit/label) with respect to full 4-architecture switch.

5 Discussion

Too many parameters in Deep Learning models? From an information theory perspective, the goal of a model is to extract as much mutual information between the labels and inputs as possible—equivalently (Section 2.4), to compress

the labels. This cannot be achieved with 2-part codes or practical network compression. With the variational code, the models do compress the data, but with a worse prediction performance: one could conclude that deep learning models that achieve the best prediction performance cannot compress the data.

In [GJ16] the authors empirically study the *degrees of freedom* of deep neural models and show it is an order of magnitude lower than the number of parameters: from an AIC perspective, the number of parameters might not necessarily be a problem for model selection.

Thanks to prequential and switch codes, we have seen that deep learning models, even with a large number of parameters, are able to compress the data: from an information theory point of view, *the number of parameters is not an obstacle to compression.*

Comprehension and Compression. Chaitin's hypothesis on the equivalence between comprehension and compression is the foundation of the Minimum Description Length principle and Solomonoff's theory of induction [Sol64]. This can also be used as a model selection procedure: select the model which compresses the most. On the contrary, in statistical learning theory, the selected model is the model which generalizes best on a test set, based, e.g., on a cross-validation procedure.

Under the strong assumption that the model parameter is identifiable (which excludes most deep learning models) and the data are actually produced with this model, the differences between model selection via compression or generalization has been well studied: MDL model selection is asymptotically equivalent to the BIC criterion [Grü07], while the leave-one-out cross-validation is equivalent to the AIC criterion [Sto77]. BIC and AIC both have desirable properties. The BIC criterion is consistent, and the AIC criterion has an optimal convergence rate. Model switching has been presented as a way to get almost the best properties of these two approaches [vdPG14].

In the practical results presented here, models which are better at generalization on new data also achieve great compression bounds with the prequential or switch codes. This is consistent with Chaitin's hypothesis, contrary to previous observations with the variational code.

Inefficiency of Variational Models for Deep Networks. Variational methods are a strong regularization for training deep neural networks, but lead to worse performance than more usual training methods.

The objective of variational methods is equivalent to minimizing a description length. On image classification, variational methods do not have good results *even for their own objective*, compared to prequential codes. This makes their relatively bad results at test time less surprising.

Understanding this observed inefficiency of variational methods in the networks above is an open problem. As stated in (13) above, the variational codelength is an upper bound for the Bayesian codelength. More precisely, with the notation above, we have:

$$L_\beta^{\mathrm{var}}(y_{1:n}|x_{1:n}) = L^{\mathrm{Bayes}}(y_{1:n}|x_{1:n}) + \mathrm{KL}\left(p_{\mathrm{Bayes}}(\theta|x_{1:n}, y_{1:n})\|\beta\right) \tag{19}$$

where $p(\theta|x_{1:n}, y_{1:n})$ is the Bayesian posterior on θ given the data. Empirically, on MNIST and CIFAR,

$$L^{\mathrm{preq}}(y_{1:n}|x_{1:n}) \ll L_\beta^{\mathrm{var}}(y_{1:n}|x_{1:n}). \tag{20}$$

The large difference between these two codelengths is the sum of three terms, all of which could explain the gap:

- The optimization of the parameters ϕ of the approximate Bayesian posterior $\beta_{\hat{\phi}}$ might be imperfect.
- The optimal distribution β^* might not approximate the posterior distribution $p(\theta|x_{1:n}, y_{1:n})$ well, leading to a large term $\mathrm{KL}(p(\theta|x_{1:n}, y_{1:n})\|\beta^*)$ in (19). This is a problem with choosing the model class for β.
- Finally, the (untractable) Bayesian codelength may be itself larger than the prequential one. This would be a problem of underfitting in parametric Bayesian inference (even on the codelength or Bayesian evidence bound itself, not only on generalization performance): in our limited experiments, incremental encoding with a parameter learned by SGD works better than with the Bayesian posterior.

Sorting these contributions is a topic for future research.

6 Conclusion

Deep learning models can represent the data *together with the model* in fewer bits than a naive encoding, despite their many parameters. This also measures the amount of mutual information between the inputs and outputs. However, variational inference, though explicitly designed to minimize such codelengths, provides very poor values in this context compared to a simple incremental (prequential) coding scheme.

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A Technical Details

We describe the technical details of the models used for the experiments in this paper. All the code will be publicly available.

A.1 Variational Learning

For Variational Learning, we used a prior $\alpha = \mathcal{N}(0, \sigma_0^2 I_d)$ with $\sigma_0 = 0.05$, chosen to optimize the compression bounds.

The chosen class of posterior was the class of multivariate gaussian distributions with diagonal covariance matrix $\{\mathcal{N}(\mu, \Sigma)\ ,\ \mu \in \mathbb{R}^d\ \Sigma \text{ diagonal}\}$. It was parametrized by $(\beta_{\mu,\rho})_{(\mu,\rho)\in\mathbb{R}^d\times\mathbb{R}^d}$, with $\sigma \in \mathbb{R}^d$ defined as $\sigma_i = \log(1 + \exp(\rho_i))$, and the covariance matrix Σ as the diagonal matrix with diagonal values $\sigma_1^2, ..., \sigma_d^2$.

We optimize the bound (9) as a function of (μ, ρ) with a gradient descent method, and estimate its values and gradient with a Monte-Carlo method. Since the prior and posteriors are gaussian, we have an explicit formula for the first part of the variational loss $\mathrm{KL}(\beta_{\mu,\rho}\|\alpha)$ [HV93]. Therefore, we can easily compute its values and gradients. For the second part

$$(\mu, \rho) \to \mathbb{E}_{\theta\sim\beta_{\mu,\rho}}\left[\sum_{i=1}^{n} -\log_2 p_\theta(y_i|x_i)\right], \tag{21}$$

we can use the following proposition [Gra11]. For any function $f : \Theta \to \mathbb{R}$, we have

$$\frac{\partial}{\partial\mu_i}\mathbb{E}_{\theta\sim\beta_{\mu,\rho}}[f(\theta)] = \mathbb{E}_{\theta\sim\beta_{\mu,\rho}}\left[\frac{\partial f}{\partial\theta_i}(\theta)\right] \tag{22}$$

$$\frac{\partial}{\partial\rho_i}\mathbb{E}_{\theta\sim\beta_{\mu,\rho}}[f(\theta)] = \frac{\partial\sigma_i}{\partial\rho_i}\cdot\mathbb{E}_{\theta\sim\beta_{\mu,\rho}}\left[\frac{\partial f}{\partial\theta_i}\cdot\frac{\theta_i-\mu_i}{\sigma_i}\right] \tag{23}$$

Therefore, we can estimate the values and gradients of (9) with a Monte-Carlo algorithm:

$$\frac{\partial}{\partial\mu_i}\mathbb{E}_{\theta\sim\beta_{\mu,\rho}}[f(\theta)] \approx \sum_{s=1}^{S}\frac{\partial f}{\partial\theta_i}(\theta^s) \tag{24}$$

$$\frac{\partial}{\partial\rho_i}\mathbb{E}_{\theta\sim\beta_{\mu,\rho}}[f(\theta)] \approx \frac{\partial\sigma_i}{\partial\rho_i}\cdot\sum_{s=1}^{S}\frac{\partial f}{\partial\theta_i}(\theta^s)\cdot\frac{\theta_i^s-\mu_i}{\sigma_i} \tag{25}$$

where $\theta^1, ..., \theta^S$ are sampled from $\beta_{\mu,\rho}$. In practice, we used $S = 1$ both for the computations of the variational loss and its gradients.

We used both convolutional and fully connected architectures, but in our experiments fully connected models were better for compression. For CIFAR and MNIST, we used fully connected networks with two hidden layers of width 256, trained with SGD, with a 0.005 learning rate and mini-batchs of size 128.

During the test phase, we sampled parameters $\hat{\theta}$ from the learned distribution β, and used the model $p_{\hat{\theta}}$ for prediction. This explains why our test accuracy on MNIST is lower than other numerical results [BCKW15], since they use for prediction the averaged model with parameters $\hat{\theta} = \mathbb{E}_{\theta\sim\beta_{m,r}}[\theta] = \mu$. But our goal was not to get the best prediction score, but to evaluate the model which was used for compression on the test set.

A.2 Prequential Learning

Prequential Learning on MNIST On MNIST, we used three different models:

1. The uniform probability over the labels.

2. A fully connected network or Multilayer Perceptron (MLP) with two hidden layers of dimension 256.

3. A VGG-like convolutional network with 8 convolutional layers with 32, 32, 64, 64, 128, 128, 256 and 256 filters respectively and max pooling operators every two convolutional layers, followed by two fully connected layers of size 256.

For the two neural networks we used Dropout with probability 0.5 between the fully connected layers, and optimized the network with the Adam algorithm with learning rate 0.001.

The successive timestep for the prequential learning $t_1, t_2, ..., t_s$ are 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384 and 32768.

For the prequential code results in Table 1, we selected the best model, which was the VGG-like network. For the switch code, we switched between the three models. For the self-switch code, we used only the VGG-like network at several epochs.

Prequential Learning on CIFAR On CIFAR, we used four different models:

1. The uniform probability over the labels.

2. A fully connected network or Multilayer Perceptron (MLP) with two hidden layers of dimension 512.

3. A convolutional network (tinyCNN) with four convolutional layers with 32 filters, and a maxpooling operator after every two convolutional layers. Then, two fully connected layers of dimension 256. We used Dropout with probability 0.5 between the fully connected layers.

4. A VGG-like network with 13 convolutional layers from [Zag15]. We trained this architecture with two learning procedures. The first one (VGGa) without batch-normalization and data augmentation, and the second one (VGGb) with both of them, as introduced in [Zag15].

We optimized the network with the Adam algorithm with learning rate 0.001.

For prequential learning, the timesteps $t_1, t_2, ..., t_s$ were: 10, 20, 40, 80, 160, 320, 640, 1280, 2560, 5120, 10240, 20480, 40960. The training results can be seen in Figure 2.

For the prequential code, all the results are in Figure 2. For the results in Table 1, we selected the best model for the prequential code, which was VGGb. For the switch code, we switched between the five models. For the self-switch code, we used only the VGGb model at several epochs.